Numerical Computation of the Large Deviation Multifractal Spectrum

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Abstract

We propose a kernel method for the estimation of the large deviation multifractal spectrum, and report on some preliminary results.

1 Introduction

Since its introduction ([14, 15, 7]), multifractal analysis (MA) has proved a valuable tool in many area of physics ([16, 17, 5, 8, 3]), signal processing ([12, 10]) and other fields. A considerable amount of mathematical work has been devoted to it ([6, 2, 9, 1, 18, 19, 13]). MA has been applied to turbulence analysis, DLA patterns investigation, random electrical networks analysis, image processing, road traffic modelling and many other topics.

The main feature of MA is the multifractal spectrum, a function which quantifies the "number" of points having the same "singularity". Thus, for instance, in a turbulent fluid flow, the multifractal spectrum describes the range of exponents characterizing energy dissipation.

One of the major drawback of MA is that the multifractal spectrum is extremely difficult to estimate on real, discrete data. For this reason, several other definitions of multifractal spectra have been proposed in the litterature for which it is easier to design estimation procedures. It should be noted however that all these definitions are not equivalent.

After some recalls, we focus in this paper on the so-called "large deviation spectrum",denoted f_g, which offers a good compromise between accuracy and difficulty of computation. We give new results that allow useful insights on the structure of f_g, and propose a kernel method for its estimation. Preliminary results where such an approach can be fully justified are given. Experimental results are presented in the last section.

2 Multifractal spectra

For simplicity, we restrict ourselves here to the one dimensional case. Let μ be a measure defined on $[0,1[$, and $\mathcal{P}$ be a sequence of partitions $\mathcal{P}_n$ of $[0,1[$, each $\mathcal{P}_n$ being made of dyadic intervals, i.e.:

$$\mathcal{P}_n = \left\{I_n^k\right\}_{0\le k<2^n}, I_n^k = [k2^{-n}, (k+1)2^{-n}[$$

(The general case of non uniform partitions will not be considered here, see for instance [13].)

For $\alpha \in \mathbb{R}$, let: $f_h(\alpha) = \dim_H\{x \in [0,1[/\alpha(x) = \alpha\}$ where $\dim_H$ denotes the Hausdorff dimension, and:

$$\alpha(x) = \lim_{n\to\infty} \frac{-\log \mu(I_n(x))}{n}$$

provided that the limit exists ($I_n(x)$ denotes the interval I_n^k containing x). Let:

$$\alpha_n^k = \frac{-\log \mu(I_n^k)}{n} \qquad f_g(\alpha) = \lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\frac{\log N_n^\varepsilon(\alpha)}{n} \qquad (1)$$

where $N_n^\varepsilon(\alpha) = \#\{\alpha_n^k/\alpha_n^k - \varepsilon \le \alpha < \alpha_n^k + \varepsilon\}$. Finally, let:

$$\tau(q) = \lim_{n\to\infty} \frac{-\log \sum_k \mu(I_n^k)^q}{n}$$

when the limit exists, and $f_l(\alpha) = \inf_q(q\alpha - \tau(q))$

f_h is called the Hausdorff multifractal spectrum of μ, and is usually the quantity of interest. It gives a "geometrical" description of the singularity structure. f_g is called the large deviation spectrum, it quantifies the "probabilistic" repartition of the singularities. Finally, f_l is called the Legendre spectrum. It was essentially introduced because it is much easier to compute than f_g and f_h. Indeed, it involves only averaged quantities, and only one limit is needed for its computation. However, its use implies a severe loss of informations since f_l is always a concave function. Although in some cases it can be proved that $f_h = f_g = f_l$, in general we only have $f_h \le f_g \le f_l$ and $f_g^{**} = f_l$, where f_g^{**} is the concave hull of f_g ([2, 1, 13]). Since f_g is in general "closer" to f_h than is f_l, it seems worth to try to find reliable estimation procedures for it. Let us take two simple examples to show why f_g is a "better" spectrum than f_l.

- Assume that we are observing a multiplicative process whose parameters may assume a finite number of distinct values in time. For instance, it could be a binomial measure [1] with weights $(m, 1-m)$ for $x \in \left[0, \frac{1}{2}\right[$, and another binomial measure with weights $(p, 1-p)$ on $\left[\frac{1}{2}, 1\right[$ In this case, the theoretical f_h and f_g spectra coincide and are not concave (see figure 2). However, f_l, which is the concave hull of f_g, will overestimate the true dimensions. In addition, f_l will not allow to detect the non stationnarity of the process.
- Assume now that the observed process is the sum of a finite number of multiplicative process, say two binomial measures. Here again, $f_h = f_g$ will not be concave (see figure 1). Computing f_l will result in an overestimation of some singularities dimensions and will not allow to detect the compound nature of the process.

These considerations motivate the search for reliable f_g spectra estimation procedures.

3 Multifractal spectrum estimations

We restrict ourselves here to f_g spectrum estimation. For f_h spectrum estimation, see [11], and for f_l spectrum [5]. f_g spectrum estimation has been considered by a number of authors ([5]). Roughly speaking, all methods involve the following steps:

1. for each n, compute all α_n^k;
2. compute $\alpha_n^{\min} = \min_k \alpha_n^k, \alpha_n^{\max} = \max_k \alpha_n^k$, and divide $[\alpha_n^{\min}, \alpha_n^{\max}]$ into N boxes
3. compute the number N_n^i of intervals I_n^k whose α_n^k falls in the i-th box, $i = 1, \ldots N$;
4. find $f(\alpha)$ by a linear regression on $(\log n, \log N_n^i)$.

Although, this "histogram method" may yield satisfactory results on strictly multiplicative cascades, it fails to estimate a good approximation in more complex situations, in particular when f_g is not a concave function. For instance, in the two situations presented above (non stationnary or compound process), the histogram method will tend to hide the non concavity of f_g. There are two types of problems with this method:

- The choice of the number of "boxes" on the α axis: this choice is ad-hoc, although it influences a lot the result. In particular, it does not take into account the ε limit in (1).
- The choice of the averaging number: when dealing with, say, a triomial measure, averaging two adjacent boxes at each step produces strong oscillations with induce errors in the regression estimation step, and thus both in the exponent and dimension computations.

As said above, this method in fact introduces an implicit dependence between n and ε through step 2. Although it is desirable, in numerical methods, to pass from two limits on n and on ε to just one limit, it is clear that the dependency between n and ε should be carefully designed.

We propose here to estimate f_g with a classical tool in density estimation: the kernel method, or more precisely the double kernel method ([4]). Such a method has been used extensively with good success in density estimations, and rather precise theorems are known that assess the quality of the results. The difficulty here is that f_g is not the density corresponding to the α's, but rather a double logarithmic normalization of this density.

Our estimation procedure has the following advantages:

- The choice of the number of boxes is no longer ad-hoc but driven by the data. The dependence on ε is made explicit. One should realize that the limit on ε plays a crucial role: if ε is chosen equal to 0, we would for instance find $f(\alpha) = 0$ almost everywhere for a binomial measure.

[1] A binomial measure $(m, 1-m)$ puts mass $m^{n\rho}(1-m)^{n(1-\rho)}$ on interval I_n^k, where ρ is the proportion of 0 in the first n digits of any x in I_n^k.

- No averaging needs to be done, since the spectrum is evaluated at only one resolution. Of course, in the general case, one needs to verify that the spectra estimated at each resolution are consistent. Such a verification should be performed carefully. However, in the cases considered here, this problem does not arise. We note here that other authors have considered one resolution estimation methods ([8]).
- The use of smooth kernels allows to obtain more regular estimates.

The starting point of the method is to realize that $N_n^\varepsilon(\alpha)$ may be re-written as the following convolution: $N_n^\varepsilon(\alpha) = 2^{n+1}\varepsilon K_\varepsilon * \rho_n(\alpha)$ where ρ_n is the density of the $(\alpha_n^k)_k$, and K_ε is the rectangular kernel: $K(x) = 1$ for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, 0 outside, and $K_\varepsilon(x) = \frac{1}{\varepsilon}K\left(f\frac{x}{\varepsilon}\right)$.

It is easily shown that K may be in fact chosen to be any compactly supported kernel. A classical problem in density estimation is the choice of an ε such that $K_\varepsilon * \rho_n$ is as close as possible to the limiting density ρ. In this setting, ε becomes a function of n. Adapting classical techniques allows us to reduce the two limits in f_g to one limit, with an optimal choice of ε. However, the first thing to check is under which conditions it is indeed possible to get rid of the limit on ε. We have the following results.

Theorem 1 *Let $f_g^\epsilon(\alpha) = \overline{\lim}_{n\to\infty}\frac{\log N_n^\varepsilon(\alpha)}{n}$. The following statements are equivalent :*

- $f_g^\epsilon(\alpha) = \sup_{x\in\mathcal{B}(\alpha,\epsilon)} f_g(x)$.
- *f_g^ϵ is a left continuous function of ϵ for positive ϵ.*

Proposition 1 *Let μ be a binomial measure with weights (m_0, m_1), and $a = |\log_2 \frac{m_1}{m_0}|$. Then, $\forall\alpha$, there exists a subsequence $s(n)$ such that $\lim_n f_{s(n)}^{\varepsilon_{s(n)}}(\alpha) = f(\alpha)$ as soon as $\varepsilon_n \geq \frac{a}{n^2}, \varepsilon_n \to 0$. Moreover, $\lim_n \sup_\alpha |f_n^{\varepsilon_n} - f(\alpha)| = 0$ as soon as $\varepsilon_n \geq \frac{a}{2n}, \varepsilon_n \to 0$.*

If we replace the rectangular kernel by a Gaussian kernel, we obtain

Proposition 2 *For a binomial measure, the optimal ε_n (in the sense of uniform convergence) for a Gaussian kernel is such that there exists $c > 0$ with $\lim_n(\varepsilon_n/c\sqrt{n}) = 1$.*

The double kernel method consists in choosing ε_n such that the estimates $f_n^{\varepsilon_n}$ and $g_n^{\varepsilon_n}$ with two different kernels are as close as possible. Let $f_n^{\varepsilon_n}$ correspond to the rectangular kernel and $g_n^{\varepsilon_n}$ correspond to the triangular kernel ($T(x) = 1-x$ for $x \in [0,1], T(x) = 0$ for $x > 1, T(-x) = T(x)$). The following propositions are valid for finite sums or lumpings of multinomial measures:

Proposition 3 *Let ε_n be a sequence that minimizes $\sup_\alpha(f_n^{\varepsilon_n}(\alpha) - g_n^{\varepsilon_n}(\alpha))$. Then $f_n^{\varepsilon_n} \to f(\alpha)$ for all α.*

Proposition 4 *Let ε_n be such that there exists $c > 0$ with $\lim_n \frac{\varepsilon_n}{cn} = 1$. Then $\lim_n \sup_\alpha(f_n^{\varepsilon_n}(\alpha) - f(\alpha)) = 0$.*

It should be noted that the double kernel method does not in general give the optimal sequence ε_n.

4 Results

The theoretical f_g along with its kernel estimate for a sum (resp. a lumping) of two binomial measures is shown in figure 1 (resp. figure 2). A slightly more complex example is shown on figure 3: it is the estimated f_g spectrum of a trinomial measure T with overlapping: the first map sends $[0,1]$ to $\left[0, \frac{1}{2}\right]$ and has weight $m, 0 < m < \frac{1}{2}$, the second map sends $[0,1]$ to $\left[\frac{1}{4}, \frac{3}{4}\right]$ and has weight $\frac{1}{2}$, and the third map sends $[0,1]$ to $\left[\frac{1}{2}, 1\right]$ and has weight $\frac{1}{2} - m$. The theoretical f_g spectrum is not known in this case. However, it is readily proven, using for instance Fourier transformation, that T is the convolution of the Lebesgue measure with a binomial measure with weights $(m, 1-m)$. In particular, all points have Hölder exponent greater than one, a fact that can be verified on figure 3. The last example deals with real data from a turbulent flow provided by G. Antar for École Polytechnique in France. The fact that the estimated f_g is concave is consistent with the hypothesis that turbulence may, in this case, be modeled as *one* multiplicative process.

5 Conclusion

This work has shown that it is possible to obtain reliable estimation procedures for the f_g spectrum in the case of non pure multiplicative processes. A lot remains to be done in order to fully justify the method in more general situations.

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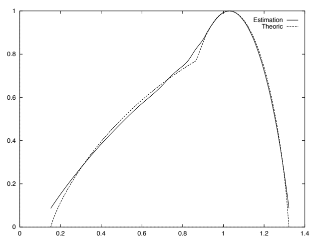

Figure 1: Theoretical (dashed) and estimated (solid) f_g spectrum for the sum of two binomial measures.

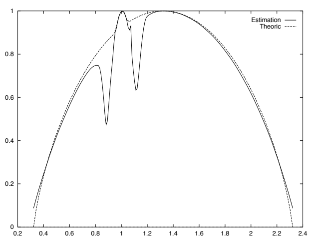

Figure 2: Theoretical (dashed) and estimated (solid) f_g spectrum for the lumping of two binomial measures.

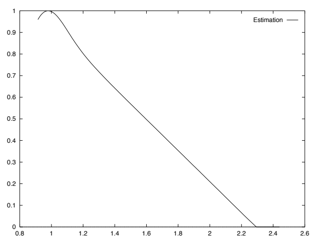

Figure 3: Estimated f_g spectrum for a trinomial measure with overlapping.

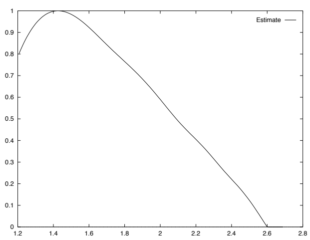

Figure 4: Estimated f_g spectrum for turbulence fluid flow.

References

[1] M. Arbeiter and N. Patzschke. Random Self-Similar Multifractals. *Advances in Mathematics*, to appear.

[2] G. Brown, G. Michon, and J. Peyrière. On the Multifractal Analysis of Measures. *Journal of Stat. Phys.*, t. 66:775–790, 1992.

[3] A. Chhabra, R. Jensen, and K. Sreenivasan. Extraction of underlying multiplicative processes from multifractals via the thermodynamic formalism. *Phys. Rev. A*, 40(8):4593–4611, 1989.

[4] L. Devroye. The double kernel method in density estimation. *Ann. Inst. Henri Poincaré*, 25(4):533–580, 1989.

[5] C.J.G. Evertsz and B.B. Mandelbrot. Multifractal Measures. In *Chaos and Fractals : New Frontiers in Science*, pages 849–881. H.-O. Peitgen, H. Jürgens and D. Saupe, Springer, New-York, 1992.

[6] K.J. Falconer. The Multifractal Spectrum of Statistically Self-Similar Measures. *Journal of Theoretical Probability*, 7(3):681–702, July 1994.

[7] U. Frisch and G. Parisi. *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, page 84. M. Ghil, R. Benzi and G. Parisi, Amsterdam (Holland), 1985.

[8] P. Grassberger, R. Badii, and A. Politi. Scaling Laws for Invariant Measures on Hyperbolic and Nonhyperbolic Atractors. *Journal of Statistical Physics*, 51(1/2):135–178, July 1988.

[9] R. Holley and E. Waymire. Multifractal dimensions and scaling exponents for strongly bounded random cascades. *Ann of App. Prob.*, 2(4):819–845, 1992.

[10] J. Lévy Véhel and J.-P. Berroir. Image Analysis through Multifractal Description. In *Fractal'93*, Londres, 1993.

[11] J. Lévy Véhel and C. Canus. Hausdorff dimension estimation and application to multifractal spectrum computation. Technical report, INRIA, 1996.

[12] J. Lévy Véhel and P. Mignot. Multifractal Segmentation of Images. *Fractals*, 2(3):371–377, 1994.

[13] Jacques Lévy Véhel and Robert Vojak. Multifractal Analysis of Choquet Capacities : Preliminary Results. *to appear in Advances in Applied Mathematics*, 1995.

[14] B.B. Mandelbrot. Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence. In *Statistical Models and Turbulence*, pages 331–351, La Jolla, California, 1972. Edited by Murray Rosenblatt and Charles Van Atta, New York, Springer. (Lecture Notes in Physics, 12).

[15] B.B. Mandelbrot. Intermittent turbulence in self similar cascades : divergence of high moments and dimension of the carrier. *J. Fluid Mech.*, 62:331–358, 1974.

[16] B.B. Mandelbrot. Multifractal measures, especially for the geophysicist. *Pure and Applied Geophysics*, 131:5–42, 1989.

[17] B.B. Mandelbrot. Limit lognormal multifractal measures. In *Frontiers of Physics : Landau Memorial Conference*, number 163, pages 309–340, (Tel Aviv, 1988), 1990. Ed. by E.A. Gostman et al, New York, Pergamon.

[18] L. Olsen. Random Geometrically Graph Directed Self-similar Multifractals. *Pitman Research Notes in Mathematics*, 1994. Series 307.

[19] R. Riedi. An improved multifractal formalism and self-similar measures. *Journal Math. Anal. Appl.*, 189, 1995.